Discrete Mathematics 14 (1976) 257–264.

SIMPLE DIRECTED TREES

R.H. JOHNSON

Department of Mathematics, California State College, Sonoma, Rohnert Park, Calif. 94928, USA

Received 6 August 1974

Those directed trees which are determined up to isomorphism by their demi-degrees are characterized.

In this note the problem of determining those graphs which are determined up to isomorphism by their demi-degrees is discussed. Following Harary such graphs are called simple and the main result contained herein is a characterization of simple trees.

All graphs discussed below are finite, directed and without multiple arcs. That is, a graph G is a pair (X, Γ) where X is a finite non-empty set and Γ is a binary relation on X. The set of vertices of G, $V(G)$, is X and the set of arcs of G, $A(G)$, is Γ. Also $\Gamma x = \{y: (x, y) \in A(G)\}$ and $\Gamma^{-1}x = \{y: (y, x) \in A(G)\}$. The set of pairs of the form $(|\Gamma x|, |\Gamma^{-1}x|) = (\deg^+ x, \deg^- x)$ for $x \in X$ is called the *demi-degrees* of G (note that for a set S, $|S|$ denotes the number of elements of S). If a set S is the demi-degrees of G, then G is said to *realize* S. Finally, if S is a set of demi-degrees realized by exactly one graph G, then both G and S are called *simple*.

The concept of a simple graph arises naturally as one tool to understand the relation of the structure of a graph to its demi-degrees since in the simple case the graph and hence its structure are determined uniquely by its demi-degrees.

The problem of characterizing simple non-directed multigraphs has been settled quite easily by Senior in [4]. Simple "ordinary" graphs (non-directed, without multiple edges) have not been characterized but simple ordinary trees have (in [3]), and we shall refer to this later.

Unless stated otherwise we shall continue to use the terminology and notation of Berge [1].

Perhaps the concept which has been the key to understanding demi-degrees is that of a "transfer".

Definition. Let G be a graph and assume x, y, z, w are distinct points of $V(G)$ such that $(x, y) = xy$ and $(z, w) = zw$ are in $A(G)$ but $(x, w) = xw$ and $(z, y) = zy$ are not in $A(G)$. A *transfer* t to G is the replacement of xy and zw by xw and zy. The new graph so obtained is denoted by tG. If in tG, xw and zy are replaced by xy and zw this transfer is denoted by t^{-1}.

It should be clear that $G = t^{-1}tG$. Further, G and tG have the same demi-degrees. More important, the converse of the latter is also true, namely, if two graphs have the same demi-degrees then they *differ* by a finite sequence of transfers. Berge [1] attempts a proof of this but his method requires transfers which may create multiple arcs (since he deletes all arcs in common at the outset). Fulkerson et al. [2] have given the proof in the undirected case. Thus for completeness a proof of the following is included.

Theorem 1. *Let G and H be graphs having the same demi-degrees. There exists a finite number of transfers $t_1, ..., t_r$ such that $H = (t_r \circ ... \circ t_1)G$.*

Proof. Assume that $V(G) = V(H) = \{x_1, ..., x_p\}$. Let A, B be the adjacency matrices of G and H respectively. Since G, H belong to the same degree sequence, we have

$$\sum_{i=1}^{p} A_{ij} = \sum_{i=1}^{p} B_{ij} = \deg^-(x_j), \quad j \in \{1, ..., p\},$$
$$(1)$$
$$\sum_{j=1}^{p} A_{ij} = \sum_{j=1}^{p} B_{ij} = \deg^+(x_i), \quad i \in \{1, ..., p\}.$$

Now let $C = A - B$. Then for any $i, j \in \{1, ..., p\}$ we have $C_{ij} \in \{-1, 0, 1\}$ and

$$\sum_{j=1}^{p} C_{ij} = \sum_{i=1}^{p} C_{ij} = 0$$

by (1) above.

If $\Sigma_{ij} |C_{ij}| = 0$, then $A = B$ and we are done. Hence, assume $\Sigma_{ij} |C_{ij}| > 0$. We now show that there is a transfer t of either G or H, say G, so that if A_t is the adjacency matrix of tG and $C^1 = A_t - B$ then

$$\sum_{ij} |C^1_{ij}| < \sum_{ij} |C_{ij}|.$$

Assuming $\Sigma_{ij}|C_{ij}| > 0$ yields immediately by (1) that there exist $i, j, k \in \{1, ..., p\}$ such that $C_{ij} = 1$, $C_{kj} = -1$. Again by (1) there is an $l \in \{1, ..., p\}$ such that $C_{kl} = 1$. Again, find $h \in \{1, ..., p\}$ such that $C_{hl} = -1$. We proceed in this fashion, constructing a sequence of the form $C_{ij}C_{kj}C_{kl}C_{hl}C_{km}$ ..., where the terms are alternately +1 and −1.

We also require in constructing the sequence that all terms but the first and the last be distinct, i.e., if C_{rs} and C_{uv} are not both end terms then $(r, s) \neq (u, v)$. Notice now that C can be regarded as the adjacency matrix of a directed graph, with vertex set $\{x_1, ..., x_p\}$, where each node has as many arcs entering as leaving. Hence any such sequence starting at the node x, if continued as far as possible, must end at x. We can then assume that the next to last term of the sequence has the form $C_{ir} = -1$.

Now consider the "rectangle" C_{ij}, C_{kj}, C_{kl} and C_{il}. There are three cases according as whether C_{il} is −1, 0, or −1.

Case 1. $C_{il} = -1$. In this case x_ix_j, x_kx_l are in $A(G)$ but not in $A(H)$ and x_kx_j, x_ix_l are in $A(H)$ but not in $A(G)$. Hence the transfer t, in G, of x_ix_j and x_kx_l for x_kx_j and x_ix_l yields a matrix $C^1 = A_t - B$, where C^1 is the same as C except the entries C^1_{ij}, C^1_{kj}, C^1_{kl}, C^1_{il} are all zero. Hence $\Sigma_{ij}|C^1_{ij}| < \Sigma_{ij}|C_{ij}|$.

Case 2: $C_{il} = 0$. If also $A_{il} = B_{il} = 0$, then the same transfer as per Case 1 can be made since $x_ix_l \notin A(G)$ and now C^1 is the same as C except $C^1_{ij} = C^1_{kj} = C^1_{kl} = 0$ and $C^1_{il} = 1$. If, on the other hand, $A_{il} = 1$, $B_{il} = +1$, then we make a transfer t in H by replacing x_ix_l and x_kx_j by x_ix_j and x_kx_l. If $C^1 = A - B_t$, then this C^1 is exactly the same as C^1 obtained in the first part of this case.

Case 3: $C_{il} = +1$. In this case we can replace the sequence $C_{ij}C_{kj}C_{kl}C_{hl} \cdots C_{ir}C_{ij}$ by the shorter sequence $C_{il}C_{hl} \ldots C_{ir}C_{il}$. We now apply the method of Case 1 and Case 2 to the new sequence to obtain the desired C^1 or a shorter sequence. Hence it is clear that we get the desired C^1 at some stage – if the sequence is shortened to five terms then the fourth term in the sequence must be −1.

Now consider C^1. If $\Sigma_{ij}|C^1_{ij}| = 0$, then $G = tH$ or $tG = H$. If $\Sigma_{ij}|C^1_{ij}| > 0$ we operate on C^1 as we did on C. It follows that at some point we get a C^n such that $\Sigma_{ij}|C^n_{ij}| = 0$ and this means that there exist transfers $t_1, ...,$ t_N, $s_1, ..., s_M$ such that $(t_N \circ ... \circ t_1)\, G = (s_M \circ ... \circ s_1)H$. But then

$$(s_1^{-1} \circ ... \circ s_M^{-1} \circ t_N \circ ... \circ t_1)G = H\,.$$

This proves the theorem.

Definition. If G is a graph then the *converse* of G, denoted by G', has $V(G') = V(G)$ but $(x, y) \in A(G')$ if and only if $(y, x) \in A(G)$ (i.e., G' is obtained from G by reversing the arcs of G).

Lemma 1. *The number of realizations of the demi-degrees of G is the same as the number of realizations of the demi-degrees of G'.*

Proof. First, if t is a transfer of G then there is an obvious corresponding transfer t' of G' and conversely. We also hope that the converse of tG is $t'G'$, i.e., $(tG)' = t'G'$. In general $[(t_1 \circ ... \circ t_r)G]' = (t'_1 \circ ... \circ t'_r)\, G'$, where the t'_i correspond to the t_i. Now note that $G \simeq tG$ if and only if $G' \simeq t'G'$ so that in general $(t_{1,1} \circ ... \circ t_{1,r})\, G \simeq (t_{2,1} \circ ... \circ t_{2,s})\, G$ if and only if $(t'_{1,1} \circ ... \circ t'_{1,r})G' \simeq (t'_{2,1} \circ ... \circ t'_{2,s})G'$. Let S, S' be the set of realizations of the demi-degrees of G and G' respectively. By Theorem 1 each element of S has the form $(t_1 \circ ... \circ t_r)G$. Finally, using the above, we note that the mapping $(t_1 \circ ... \circ t_r)G \to (t'_1 \circ ... \circ t'_r)G'$ is a one-to-one onto map of S into S'.

Corollary. *G is simple if and only if G' is simple.*

We now turn to the consideration of simple trees. We recall that a chain $[x_0 ... x_n]$ of G is a collection of arcs where for each $i = 1, ..., n$ exactly one of the arcs (x_{i-1}, x_i), (x_i, x_{i-1}) is in the collection and no others are in the collection. Also the length of the chain $[x_0 ... x_n]$ is n, and a simple chain uses no arc twice.

Lemma 2. *If T is a tree containing a simple chain of length five or greater, then T is not simple.*

Proof. The idea of the proof is to find a transfer t which creates a cycle so that $T \not\simeq tT$, since tT is not a tree. Let $[x_0 ... x_5]$ be a simple chain of length five.

Case 1. Assume $x_1 \in \Gamma x_0$ and $x_5 \in \Gamma x_4$ or $x_0 \in \Gamma x_1$ and $x_4 \in \Gamma x_5$. If the former is the case we transfer x_0x_1 and x_4x_5 for x_4x_1 and x_0x_5, creating a cycle of length four. A similar transformation has the same effect in the other possibility, or (better) one can apply Lemma 1.

Case 2. Assume $x_0 \in \Gamma x_1$ and $x_5 \in \Gamma x_4$ or $x_1 \in \Gamma x_0$ and $x_4 \in \Gamma x_5$. Again for either possibility the argument is the same so assume the latter. If $x_2 \in \Gamma x_1$ then there are two subcases according as $x_4 \in \Gamma x_3$ or $x_3 \in \Gamma x_4$.

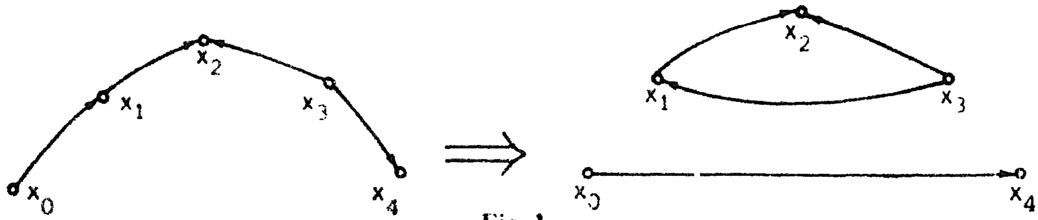

Fig. 1.

If $x_4 \in \Gamma x_3$, replace $x_3 x_4$ and $x_0 x_1$ by $x_3 x_1$, $x_0 x_4$, getting a triangle. If $x_3 \in \Gamma x_4$ then, assuming $x_2 x_3 \in A(G)$, transfer $x_0 x_1$ and $x_2 x_3$ for $x_2 x_1$ and $x_0 x_3$, getting a cycle of length two. If $x_3 x_2 \in A(G)$ we proceed in a similar fashion. We have been assuming $x_2 \in \Gamma x_1$ and if $x_1 \in \Gamma x_2$ a like argument can be used.

Finally, if the tree has a chain of length greater than five, then it clearly has a path of length five, so that the lemma is obtained.

Thus, simple trees have chains of length of at most four. What must they look like? Suppose that a tree T has a simple chain $C = [x_0 x_1 x_2 x_3 x_4]$.

Assume that G has three arcs in the "same direction". For example, suppose C consists of $x_0 x_1$, $x_1 x_2$, $x_3 x_2$, $x_3 x_4$. The transfer of $x_3 x_4$ and $x_0 x_1$ for $x_0 x_4$ and $x_3 x_1$ yields a triangle (see Fig. 1).

Similar transfers can be made in all other cases in which C has at least three arcs in the "same direction".

Finally, if C has two consecutive arcs in the "same direction", then it can be verified that one can either make a transfer to get three arcs in the same direction or to get a triangle. This leads to the following lemma.

Lemma 3. *If a simple tree has a chain $C = [x_0 x_1 x_2 x_3 x_4]$, then C must have one of the following forms*:

(a) $x_1 \in \Gamma x_0$, $x_1 \in \Gamma x_2$, $x_3 \in \Gamma x_2$, $x_3 \in \Gamma x_4$;

(b) $x_0 \in \Gamma x_1$, $x_2 \in \Gamma x_1$, $x_2 \in \Gamma x_3$, $x_4 \in \Gamma x_3$.

Proof. The lemma follows directly from the above remarks since (a) and (b) are the only possibilities that were not eliminated.

Now let T be a tree having a simple chain $C = [x_0 x_1 x_2 x_3 x_4]$ of type (a) or (b) – to be specific, type (a). Suppose that $x \in V(T) - \{x_0, x_1, x_2, x_3, x_4\}$. It is easily verified that if $x_1 \in \Gamma x$ or $x_3 \in \Gamma x$ then x has exactly one neighbor, i.e., $|\Gamma x \cap \Gamma^{-1} x| = 1$. Also, by Lemma 2, we cannot have either $x \in \Gamma x_1$ or $x \in \Gamma x_3$. Again, by Lemma 1 it follows that $x \notin \Gamma x_0$, $x_0 \notin \Gamma x$, $x \notin \Gamma x_4$, and $x_4 \notin \Gamma x$. Assuming now that x is adjacent to x_2

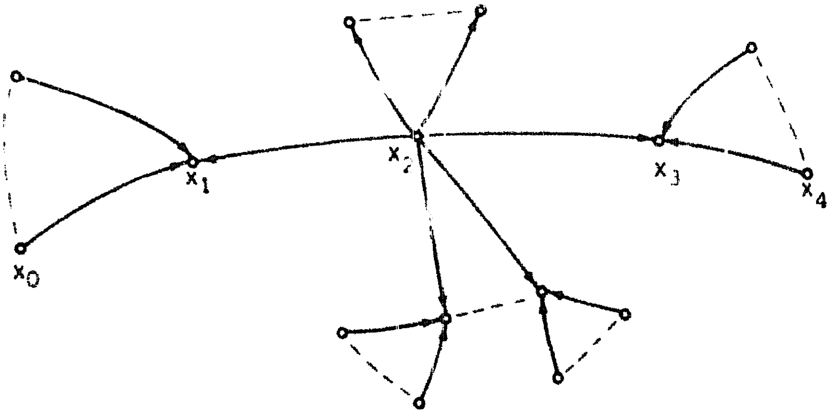

Fig. 2.

we have either $x_2 \in \Gamma x$ or $x \in \Gamma x_2$. If $x_2 \in \Gamma x$ then it can be verified that no other point is adjacent to x. If $x \in \Gamma x_2$ then there may be a point x^1 with $x \in \Gamma x^1$ but in this case we must have $\Gamma x^1 = \{x\}$, $\Gamma x = \emptyset$, and $\Gamma^{-1} x^1 = \emptyset$. Hence we are now in a position to say that if T is simple with a chain of type (a), it must lie among trees of the form in Fig. 2.

Or more generally the above figure can be re-drawn in the fashion of Fig. 3.

Definition. A graph of the type in Fig. 3 or the converse of such a graph (note that any of r, s and t may be zero) is called a *di-Giap graph.*

Theorem 2. *A tree with a chain of length four or more is simple if and only if it is a di-Giap graph.*

Proof. Using Lemma 1 and taking the converse of those graphs which

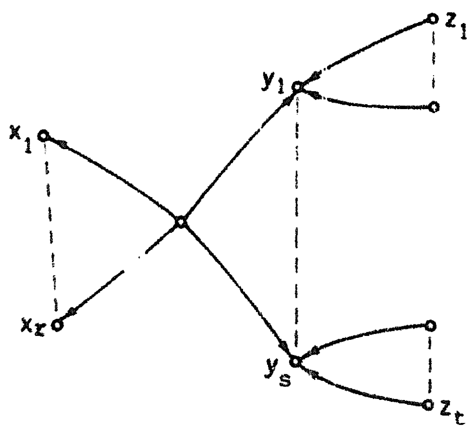

Fig. 3.

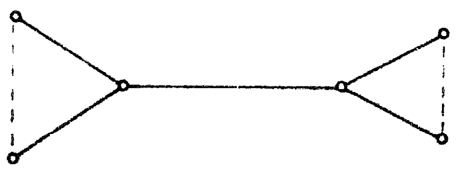

Fig. 4.

contain a chain of type (a) it follows that those graphs which contain a chain of type (b) and are simple are among converses of the kind of graphs found in Fig. 3. Further, if a simple tree has no path of length greater than three then it must be a di-Giap graph with one or more of r, s and t (in Fig. 3) equal to zero. Thus it now follows from the remarks preceding the definition of di-Giap graph that any simple tree must be of the type in Fig. 3 or its converse, i.e., any simple tree is a di-Giap graph.

For the converse it is trivial that any transfer of a di-Giap graph yields an isomorphic copy. In the only transfers defined merely interchange pendant vertices. This proves the theorem.

For completeness we note that the trees shown in Fig. 5 are the simple trees with chains of length one, two or three.

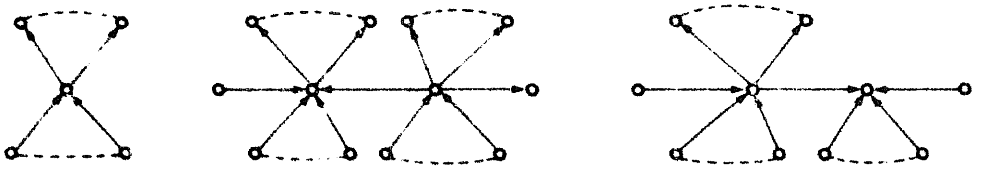

Fig. 5.

It is instructive to compare the above results with simple undirected trees considered first as multigraphs and secondly as "ordinary" graphs. The multigraph case has been easily disposed of by Senior in [4]; the only multigraph which is a simple tree on four or more vertices is a star graph.

As for "ordinary" simple trees it is shown in [3] that all simple trees (Giap graphs) have the form shown in Fig. 4.

It is clear that Giap and di-Giap graphs have an obvious similarity. In fact, the pattern of proofs in both cases is similar except that an "ordinary" simple tree cannot have a path of length greater than three.

References

[1] C. Berge, The Theory of Graphs and Its Applications (Methuen, London, 1962).
[2] Fulkerson, Hoffman and McAndrew, Some properties of graphs with multiple edges, Canad. J. Math. 17 (1965) 166–177.
[3] R.H. Johnson, Degree sequences, Ph.D. Thesis. VPI&SU, 1973.
[4] J.K. Senior, Partitions and their representative graphs, Am. J. Math. 73 (1951) 663–689.